Orthogonal Frequency Division Multiplexing (OFDM)

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1 Introduction - Orthogonality, Frequency Multiplexing

OFDM is a method of dividing a given bandwidth of frequency into smaller chunks, each of which can be independently and digitally modulated (e.g. BPSK, QPSK, QAM, ...[digital modulation techniques shall be covered in different tutorial]). It is known for being one of the most bandwidth efficient methods in communication system, which is why it has been implemented in 4G-LTE (4th generation - Long Term Evolution) and WiMax.

A defining characteristic of OFDM is the orthogonality of each of carriers. This feature (discussed below) allows for sending multiple modulated carriers at the same time without having to worry about mutual interference.

2 Complex Numbers and Euler’s Identity

Detailed analysis of complex numbers will be done in later tutorials should I end up having to use them quite a lot. For now, a basic introduction of complex numbers along with one of the most celebrated equalities in complex number theory, the Euler\(^1\) identity, shall suffice.

Complex numbers are one of the most important things in signal processing that you’ll almost never get to use in real applications (not directly, at least). However, they make mathematical analysis of signals easier by reducing most operations down to additions and subtractions.

A complex number is usually written in two forms - the rectangular form, \(a+jb\), or the complex phasor form \(ce^{j\omega}\), where \(a,b,c, \omega\) are real numbers and \(j=\sqrt{-1}\). 

\(a\) and \(b\) are also called the real and imaginary parts, and \(c\) and \(\omega\), magnitude and phase, respectively. In most real life applications, the real and imaginary parts are used as two independent streams of information that can be processed simultaneously at sending and receiving end. The Euler’s identity is given as:

\[e^{j\omega} = \cos(\omega) + j\sin(\omega)\]

Now this form is more applicable to wireless communication as each of the

\(^{1}\)Leonhard Euler (1707 - 1783) - Swiss math genius
cos and sin terms can be seen as independent carriers that can be modulated and transmitted concurrently.

3 FFT/IFFT

The Fast Fourier Transform (FFT) and Inverse Fast Fourier Transform (IFFT) are but just much faster ways of calculating the Discrete Time Fourier Transform (DTFT) of a discrete signal. The DTFT, in its most fundamental (and applicable to our case) form, is used to obtain the frequency content of a discrete signal that is sampled at some frequency $f_s$. Consider the following single-tone (single frequency) signal of 25Hz.

The figure on the top is the continuous version of the wave, while the bottom picture depicts the same continuous wave sampled at discrete times ($1/f_s$ to be exact). The DTFT, when performed on the discrete input, will output the following result (ideally).
Note that we have a spike at around 25Hz, and that is exactly what FFTs do - they give us what frequencies are present in a signal. Just take care of adjusting your sampling frequency and number of sample points taken when calculating your FFT during simulation. We’re now ready to do some mathematical analysis of the FFT/IFFT as it will make it easier to understand why OFDM works. Remember, an OFDM system does IFFT at the input, and FFT at the output.

The DTFT of a discrete time signal of length \( N \) is given as:

\[
X[k] = \sum_{n=1}^{N-1} x[n] e^{-j2\pi kn/N} \tag{1}
\]

Let’s deconstruct this equation. The \( k^{th} \) output sample, \( X[k] \), of a discrete time signal \( x[n] \), can be obtained by multiplying a sinusoid of frequency \( k \) Hz with the sampled input and summing the whole thing. Remember from your basic calculus that integrating (which is the same as adding except it’s done on a bunch of infinitesimally small portions) of two sinusoids that either do not have the same frequency or are out of phase produces a value lower that that you’d get for those when the two sinusoids have the same frequency. With this in mind, now look at equation (1) one more time. Each \( k \) in the output of the FFT is a frequency value. So for each \( k \) value, we’re generating a sinusoid (remember Euler’s relation discussed above) of that frequency, multiplying that with the input signal, and adding everything. This will give us larger values at those \( k \) (frequency) values where the input discrete signal also happens to have a sinusoidal component of the same frequency, and lower values where otherwise.

Our focus here, however, is not to use the FFT as described above, but to take advantage of its operations to modulate sinusoids of various frequencies and transmit them simultaneously (by adding them). The topic of various digital modulation techniques shall be treated in another document all together, but now, it suffices to say that modulation simply changes the amplitude and/or phase of a carrier wave (changing the frequency is also possible but is a technique not well suited for bandwidth efficient applications and is not employed in OFDM systems). Instead of performing a straight up IFFT, we will separate the input discrete signal into \( N \) blocks and use each block to multiply \( N \) different orthogonal sinusoids (discussed below) simultaneously, which are then added and transmitted. In additions to its the efficiency of the IFFT in doing fast multiplications, another important factor for using IFFT is that there exist various hardware and software solution out there which we can use directly to achieve this.

### 4 Carrier Orthogonality

In its simplest sense, orthogonality refers to how much two signals of different frequencies tend to interfere with each other. In other words, it’s possible to completely recover two (or more) orthogonal components when the received signal is a composition of them. It is also possible to independently modulate each
of these carriers with one (or multiple) of various types of modulation schemes and still be able to separate them. Let’s throw in some math here. Two signals, $s_1(t)$ and $s_2(t)$ are said to be orthogonal if their dot product is identically zero. On a signal constellation diagram, they would be the label of each of the axes because it’s a combination of those waveform that we will be playing with (I’ll have another tutorial on signal constellations and how each modulation techniques looks like on such diagrams). Let’s give a more formal definition of orthogonality. Orthogonal signals satisfy the following relation:

$$\int_a^b s_1(t)s_2(t) \, dt = 0$$

A perfect example:

$$\int_0^T \sin(\omega t)\cos(\omega t) \, dt = 0$$

The integral form of orthogonality criterion is more applicable to our case as it is actually the operations we will be carrying out at the receiver side of the communication channel. What the FFT essentially doing is re-multiplying the received signal with a sinusoid of same frequencies as used in the transmission side. Those components that are not modulated on the right carrier frequency are zeroed out.

Looking at the above picture, you can see that wherever the magnitude of any component is at its highest, there is zero contribution from any of the others. Such are the characteristics of orthogonal waveforms.

5 OFDM Channel Performance

We’ll set aside detailed analysis of channel performance such as signal-to-noise ratio (SNR), data rates, ..., and focus more on where and why you’d want to implement an OFDM system.

Fading Channels

When a signal is received at an antenna, the antenna does not just see once received waveform but multiple ones from reflections. Some of these reflections,
being waveforms like any other, could combine destructively or constructively. This is also why sometimes you can see significant increase in power received when you move as far as only a few feet from where you had been experiencing bad reception. Fading is just a term used to describe this phenomenon. There are four types of fading - slow and fast, and flat and frequency selective. Slow and fast refer to the rate of fading encountered per unit time. A more correct definition of the terms involves some more technical knowledge of channel coherent time and delay constraints but we won’t dive too deep into this now. But suffice it to say that fast fades happen rater quickly from time to time. Flat and frequency selective fades refer to how the fading affects different frequency components. Flat frequency fading happens when all frequencies experience the magnitude of fading, while in frequency selective fading, some frequencies are attenuated more than others.

So, what’s that got to do with OFDM? Well, OFDM, by chopping down the frequency spectrum into multiple (orthogonal) ones, gives you multiple flat fading channels. Now you might say, isn’t that still bad? Well, it depends. Flat fadings can be combated against much more easily than frequency selective fading - simply do more power amplification at the transmitter or receiver side. If you can do that, you can more or less have a constant amplification over all the frequency components encountered throughout the flat fading.

6 Bringing it All Together

The following two diagrams depict the major operations that take place at the transmitter and receiver side of an OFDM system (image courtesy: wikipedia)
6.1 The blocks

6.1.1 The Re and Im Paths

You can effectively double your data rate by simultaneously modulating two carriers that are out of phase by 90 degrees since they do not interfere with each other (i.e., they are orthogonal). In this case, $f_c$ will be some form of $A\cos(\omega_c t)$. The 90 degree phase shift effectively changes the lower carrier into $A\sin(\omega_c t)$. The multiplication at this stage is required to take the FFT frequency to higher levels (RF levels) suitable for transmission. This happens due to the fact that multiplication of two sinusoids shifts the frequency spectrum of the original sinusoids to their sum and difference. For example, if we multiply $\cos(at)$ and $\cos(bt)$, whose FFT would show spikes at angular frequencies $a$ and $b$ (also $-a$ and $-b$), would now be shifted to $a+b$ and $a-b$ (also $-(a+b)$ and $-(a-b)$). [check your trig identities. I’ll have more to say on these on a later discussion about Quadrature Amplitude Modulation (QAM) and other digital modulation techniques] As to why Re(al) and Im(aginary), check back to the Euler identities discussed above and you’ll quickly notice why they’re termed as such.

6.1.2 Constellation Mapper/Symbol Detection

When you modulate symbols (group of bits) on to the carrier, you’d want to make sure that there is less probability of erroneously decoding each symbol. Therefore, one of the ways to achieve this is to make sure that at most one bit changes between two consecutive symbols when you modulate your carriers. This way, when there is a burst error (error that affects a chunk of consecutive bits but leaves the rest unchanged) happens, the receiver can intelligently deduce what must have been sent even though not all the bits are present in the symbol. This saves us the trouble of having to resend that particular symbol. The detector does the reverse of this - having received the demodulated waveform that has passed through the FFT, it decides on the most likely symbol encoded by the amplitude (phase) of each subcarrier.
7 Conclusion/Future Work

You should now have a good background to start looking into more advanced techniques in OFDM based communication systems. Future related topics shall include digital modulation schemes used in OFDM and other digital communication techniques, their implementation using FPGA (by way of VHDL), as well as more theoretical short notes on topics that have to do with communication systems in general.

8 Octave(Matlab) Codes

8.1 FFT Calculation

```matlab
fs = 1000;
sample_length = 1000;
sample_time = 1/fs;
t = [0:sample_length-1]*sample_time;  % assume 1 second length in total
x = sin(2*pi*25*t);  % 25Hz signal
num_fft = 2^nextpow2(sample_length);
fft_out = fft(x,num_fft)/sample_length;
frequencies = fs/2*linspace(0,1,num_fft/2+1);
plot(frequencies, 2*abs(fft_out(1:num_fft/2+1)));```

```bash```